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Resonant Collisions in Four Dimensional Reversible Maps: A Description of Scenarios.

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Abstract: We define a resonant collision of order $k(\geq 1)$ in a family of four-dimensional reversible maps. For any specified k, the bifurcation *secnario* is the collection of the different possible *types* of bifurcation of a symmetric fixed point that may be encountered through various choices of parameters describing the family of maps under consideration. We adopt a perturbative approach, coupled with numerical iterations around orbits obtained perturbatively, to explore phase space structures in the immediate vicinity of a resonant collision and thereby to obtain a description of the possible scenarios for different values of k. The phase space structures typically involve bifurcating periodic orbits, families of invariant curves, and tori, and present interesting possibiliries, especially around the 'secondary bifurcations' of the periodic orbits (see below). Based on the results of the perturbative and numerical approach we conjecture that three distinct scenarios are involved for the cases $k = 2, 3, 4$ respectively, while there exists a fourth distinctive scenario common to all $k > 4$, and we present what we believe to be a reasonably exhaustive description of these scenarios. The case $k = 1$ involves bifurcations of fixed points rather than of periodic orbits, and has been investigated numerically in a previous paper.

I. INTRODUCTION.

Resonances in 4D symplectic and reversible maps occurring through the collision of a quartet of multipliers on the unit circle and their subsequent departure from the circle are, in a sense, counterparts of corresponding resonances in 4D Hamiltonian and reversible flows.

In the context of Hamiltonian flows a resonance of this type involves the collision on the imaginary axis of a quartet of eigenvalues associated with a fixed point, and their subsequent departure therefrom, and is termed the $1 : -1$ resonance, where the minus sign indicates that the pairs of eigenvalues involved are of opposite Krein signature [1]. An alternate nomenclature is 1:1 non-semisimple resonance (as opposed to the 1:1 semisimple resonance where the eigenvalues do not leave the imaginary axis). There is a long and distinguished history of investigations on the bifurcation associated with this resonance, originating in the restricted three-body problem ([2], see [3] for details and for historical notes), and this bifurcation has come to be known as the Hamiltonian Hopf bifurcation [3]. It is encountered in the vicinity of the Routh's critical mass ratio in the restricted three body problem [4], as well as in a spinning 'orthogonal' double pendulum [5], and in an atom placed in a rotating electric field [6].

Analogous resonance and bifurcation phenomena are encountered in reversible flows, occurring in the theory of non-linear oscillations ([7,8] ; see references therein). While the Krein theory is not applicable to non-Hamiltonian reversible flows, still these flows share many characteristic features of Hamiltonian ones close to symmetric fixed points and periodic orbits (see [7,8] for details).

In this context, it is useful to look into analogous resonances in 4D symplectic and reversible *maps* involving a quartet of multipliers of a fixed point on the unit circle where, generically speaking, results resembling those in flows are obtained. However, 4D maps are, truly speaking, representatives of dynamics of 6D flows, and this shows

up in an essential complication in collisions in 4D maps as compared to 4D flows : in case of maps one has to distinguish between the *rational* and *irrational* collisions.

While a collision itself is a resonance phenomenon, a rational collision indicates the presence of an additional resonance, and may be termed a 'resonant' collision to distinguish it from an irrational or 'non-resonant' collision, the latter being generic in the measure-theoretic sense.

Irrational collisions in 4D reversible maps have been studied in [9,10,11]. In the Hamiltonian setting they have been investigated in [12] with rigorous results. Rational collisions at $\pm i$ on the unit circle, on the other hand, have been analysed in [13] for sympletic maps, while a rational collision at $\pm i$ for a certain family of reversible maps analysed in [14] revealed features quite similar to the symplectic $\pm i$ collision (see also [15]). Interesting numerical results on bifurcations near rational and irrational collisions are to be found in [16].

A collision of a quartet of multipliers at $e^{\pm i\phi}$ (for a fixed point of a symplectic map or a symmetric fixed point of a reversible map), with $\phi = 2\pi/k$ (k positive integer) will be termed a resonant collision of order k.

While a collision of multipliers at an irrational angle is generally accompanied by the bifurcation of families of invariant curves, resonant (or rational) collisions are typically characterised by bifurcations of periodic orbits.

In this paper we present perturbation-theoretic and numerical results for resonant collisions of order k ($k = 2, 3, \cdots, 6$), from which we shall see that each of the cases $k = 2, 3, 4$ involves a characteristic bifurcation 'scenario' (see below) while the cases $k > 4$ present a common but distinct scenario. In a sense, the situation resembles the bifurcation of periodic orbits in 2D symplectic maps where resonances of order k are seen to lead to different scenarios for k=2,3 and 4 while there occurs a fourth and distinct scenario for any $k > 4$ [4, 18].

As mentioned above, the bifurcation of period-4 orbits in 4D symplectic and reversible maps has been studied in the literature in some detail [13,14]. While the dynamics in phase space around the periodic orbits was not explored in these works, preliminary results in this regard were presented in [15] where families of invariant curves around the bifurcating period-4 orbits were investigated. An interesting feature in this context is the so called 'secondary bifurcation' [13,15] whereby elliptic invariant sets involved in a rational angle bifurcation change stability causing a further change in local phase space structure. As will be seen in the following, this feature occurs commonly in the vicinity of resonant collisions of other orders as well.

We plan this paper as follows :

In section II we specify the family of maps to be studied and explain a few introductory notions. Sections III, IV and V deal with resonant collisions of order 2, 3 and 4 respectively, the first and the last of these being brief partial summaries, with necessary elaborations in a few instances, of results of ref. [15]. Section VI includes a few preliminary results on resonant collisions of orders 5 and 6. Section VII contains a synopsis and concluding remarks.

II. RESONANT COLLISIONS IN TWO-PARAMETER FAMILIES OF 4D REVERSIBLE MAPS.

In this paper we consider situations involving two-parameter families of 4D reversible maps $A_{q,\epsilon}$ such that :

(i) $A_{q,\epsilon}$ has a symmetric fixed point at the origin for each q, ϵ in some domain D of the parameter space (which is 2D, involving parameters q, ϵ) containing the point $q = 0$ and $\epsilon = 0$;

(ii) $\bar{A}_{0,0}$ has eigenvalues $e^{\pm \frac{2\pi i}{k}}$ (k=integer≥ 2) with each eigenvalue repeated twice (we call it a resonant collision of order k) or an eigenvalue -1 with multiplicity 4 (we call

it a resonant collision of order 2).

[Remark : For definitions of 'reversible map' and 'symmetric fixed point', see ref.s [9],[10] ; here $\bar{A}$ denotes the linearisation of A at the fixed point at the origin, the relevant parameter values being indicated through subscripts.]

(iii) for any q, ϵ in the domain D, other than (0,0) the linearisation $\bar{A}_{q,\epsilon}$ at the origin has distinct eigenvalues.

(iv) the Jordan block structure of $\bar{A}_{0,0}$ contains either a real Jordan block of order 4 ($k = 2$) or two complex Jordan blocks each of order 2 ($k > 2$).

It will be seen from the following that for each family of maps of the above type, the parameter space is typically divided into several regions by two curves as shown in fig.1 (these parameter regions have been marked for future reference as $P1a$, $P1b$, $P2$, and $P3$), where the parametrisation has been chosen in such a way that the curves are represented by the equations

$$\epsilon = 0, \quad q^2 + \epsilon = 0 \qquad (1a, b),$$

these curves having a tangency at (0,0). The local structure of the phase space around the fixed point is distinct in each of these regions. When any one of the boundary curves is crossed at a point other than (0,0) in the parameter space the change in the local phase space corresponds to a codimension-1 bifurcation. The point (0,0) in the parameter space, where two boundary curves meet, is degenerate, and the associated bifurcation is of codimension 2.

The local phase space structure in the vicinity of the fixed point for any $A_{q,\epsilon}$ is essentially described in terms of stability type of the fixed point and the existence and stability type of the bifurcating periodic points (a periodic point is said to bifurcate from the fixed point if it tends to and merges with the fixed point as some particular point in the parameter space is approached). Additionally, a description of families of invariant curves and tori around the periodic point may have to be included for a reasonably

complete description of the local phase space structure. Depending on the nonlinear terms, each family will be found to involve a certain number of *types* of changes in the local phase space structure as the different regions around the point (0,0) in the parameter space are approached from one another. The possible different types depend on the order of the resonant collision. Thus we get different *scenarios* of bifurcations, each scenario involving its own types of change in the local phase space structure. As already stated, the situations with $k = 2, 3$ and 4 are found to correspond to three distinct scenarios and there is a fourth scenario which is common to the situations involving all resonant collisions of order $k \geq 5$. We describe below the different scenarios obtained from our analysis. However, the description is not complete since each scenario typically involves certain degenerate types which have been left outside the purview of our investigations. These degenerate types are described by normal forms involving higher degree terms compared to the normal forms of the nondegenerate cases. While we have not attempted a normal form analysis of the resonances in the present work we conjecture that, *modulo* the degenerate cases, the scenarios indicated above do exhaust the typical behaviours of families of 4D reversible maps satisfying criteria (i)-(iv) above.

III. SECOND ORDER RESONANT COLLISIONS AND PERIOD-2 ORBITS

In order to investigate the second order resonant collision i.e., a collision near the point -1 on the unit circle, we consider the two parameter family of maps $A_{q,\epsilon}$, described in the form of the following fourth order difference equation :

$$x_{n+2} + x_{n-2} - 2(q-2)(x_{n+1} + x_{n-1}) + (q^2 - 4q + 6 + \epsilon)x_n = \beta x_n^3 \qquad (2),$$

where $\epsilon,\ q$ are small bifurcation parameters and $\beta \neq 0$ is a control parameter. The case $\beta = 0$ is degenerate and requires higher degree nonlinear terms for its description. We

consider only the non-degenerate case and normalise β to $\mid \beta \mid= 1$ through a suitable scale transformation. The inclusion of quadratic terms in the right hand side of eq. (2) does not lead to a distinct bifurcation scenario and so, for the sake of simplicity, we omit such terms.

A typical period-2 orbit of the map (2) is of the form

$$x_n = (-1)^n b_0 \qquad (3a),$$

with

$$b_0^2 = \frac{\epsilon + q^2}{\beta} \qquad (3b),$$

It follows from eqn.(3b) that a period-2 orbit can bifurcate from the fixed point in two ways, and the corresponding bifurcation diagrams are presented in fig.s 2-3, each type being associated with specific signatures of the control parameter β.

In each figure the horizontal line depicts the variation of q for fixed ϵ, while the distance of each point of the bifurcating branch from the q-axis depicts the amplitude of the orbit for the given q and ϵ. The horizontal q-axis is drawn as a solid (broken) line to indicate that the symmetric fixed point about which the periodic orbit is bifurcating, is linearly stable (unstable). Similarly, the branch, or portions thereof, denoted with solid lines (broken lines) depict linearly stable (unstable) orbits – anticipating the stability results presented below.

The linear stability analysis of the bifurcating period-2 orbit is based on the variational equation, obtained by linearising eqn.(2), around a period-2 orbit $\bar{x}_n$:

$$\xi_{n+2}+\xi_{n-2}-2(q-2)(\xi_{n+1}+\xi_{n-1})+(q^2-4q+6+\epsilon)\xi_n = 3\beta\bar{X}_n^2\xi_n \quad (\xi_n \equiv X_n-\bar{X}_n) \qquad (4).$$

In general, for an arbitrary map and for any arbitrary period-two orbit, the variational equation would be a linear difference equation with period-two coefficients, and its general solution would be made up of basic solutions of the Floquet form

$$\xi_n = \lambda^n \zeta_0 \qquad (5),$$

where λ is a quasi-multiplier (see, for an explanation, ref. [17]). In the present case, eqn.s 3(a,b),(4) imply that the period-two coefficients are just constants, but we still assume a solution of the form (5) to illustrate the procedure in general. Substituting eqn.(5) in (4) and equating the coefficients of λ^n on both sides, we get

$$(\lambda + \frac{1}{\lambda}) = -2 + q \pm \sqrt{2\epsilon + 3q^2} \qquad (6).$$

It is to be noted that eqn.(6) provide us with only one quartet of quasi-multipliers, while, in general, there should occur q number of quartets for a q-periodic point of a 4D reversible map (see ref. [17]). This is due to the special form of the variational equation indicated above, and the second quartet of quasi-multipliers in the present case can be obtained just by multiplying the multipliers belonging to the first quartet with $e^{i\pi}$.

The Floquet multipliers of the period-2 orbits are $\Lambda_i = \lambda_i^2 \ \ (i = 1, ..4)$ and can, in principle, correspond to any one of the four dispositions in the complex plane shown in fig.4 (these dispositions are marked $R1$ through $R4$ for easy reference).

Based on these results, we find that there arise two distinct bifurcation *types* (*vide* ref. [15]) which we describe below.

Type I bifurcation : $\beta > 0$.

A period-2 orbit exists on the subthreshold side, $\epsilon < 0$, for $q > \sqrt{-\epsilon}$ (region $P1a$ of parameter space, refer to fig. 1) and $q < -\sqrt{-\epsilon}$ ($P1b$), and is unstable (disposition $R3$ of fig.4). It merges with the fixed point at $\epsilon = 0, \ q = 0$ and, on the superthreshold side $\epsilon > 0$ (region $P3$ of fig.1), gets detached from the fixed point with the same type of instability ($R3$).

Type II bifurcation : $\beta < 0$.

For $\epsilon < 0$, a period-2 orbit exists if $-\sqrt{-\epsilon} < q < \sqrt{-\epsilon}$ (region $P2$ of fig.1). From eqn.(6) it follows that the period-2 orbit is unstable(with disposition $R3$ of multipliers) for $q < -\sqrt{-\frac{2\epsilon}{3}}$, unstable (with disposition $R2$) for $-\sqrt{-\frac{2\epsilon}{3}} < q < \sqrt{-\frac{2\epsilon}{3}}$, and stable

(disposition $R1$) for $q > \sqrt{-\frac{2\epsilon}{3}}$. Further, in this type of bifurcation, there exist no period-2 orbits for $\epsilon > 0$.

The secondary bifurcation

We see from above that the period-2 orbit appearing in type II bifurcation undergoes a transition from stable ($R1$) to unstable ($R2$) configuration through a *secondary bifurcation* at

$$q = q_0 \equiv \sqrt{-\frac{2\epsilon}{3}} \qquad (7).$$

The transition is characterised by the fact that the quasi-multipliers of the period-2 orbit collide pairwise at $e^{i(\pi \pm \theta_0)}$ when $q = q_0$, where

$$4sin^2\frac{\theta_0}{2} = \sqrt{\frac{2 \mid \epsilon \mid}{3}} \qquad (8).$$

This 'secondary collision' involves the bifurcation of *invariant curves* in the vicinity of the period-2 orbit.

Close to the secondary collision, the nonlinear terms become relevant, and the deviation $\xi_n = X_n - \bar{X}_n$ of a typical orbit from the period-2 orbit under consideration is given by the nonlinear difference equation

$$\xi_{n+2} + \xi_{n-2} - 2(q-2)(\xi_{n+1} + \xi_{n-1}) + (q^2 - 4q + 6 + \epsilon)\xi_n = \beta[3\bar{X}_n^2\xi_n + 3\bar{X}_n\xi_n^2 + \xi_n^3] \quad (9).$$

and, following [15], we seek solutions of the form

$$\xi_n = A_n + (B_n e^{in\phi} + B_n^* e^{-in\phi}) + (C_n e^{2in\phi} + C_n^* e^{-2in\phi}) + \cdots \qquad (10).$$

Note that the coefficients in eqn.(10) being period-2 in nature, the coefficients A_n, B_n, $C_n, \cdots$ will also be of period 2, say,

$$A_n = a + A(-1)^n \qquad (11a),$$

$$B_n = b + B(-1)^n \qquad (11b),$$

$$C_n = c + C(-1)^n \qquad (11c),$$

....

where $a, \cdots, C, \cdots$ are constants to be determined. In eqn.(10), ϕ is a rotation angle close to the collision angle $\pi + \theta_0$ (this is one of the conjugate angles at which the roots λ_i $(i = 1, \cdots, 4)$ collide), say,

$$\phi = \pi + \theta_0 + \hat{\phi} \qquad (12),$$

with $\mid \hat{\phi} \mid << \theta_0$. Different values of $\hat{\phi}$ would lead to a family of invariant curves organised in islands around the period-2 orbit.

Substituting eqn.(10) in eqn.(11) and equating the coefficients of 1, $e^{in\phi}$, $e^{2in\phi}$,.... terms respectively one gets a system of equations in A_n, B_n, $C_n, \cdots$. Substituting eqns.(11 a,b,c) in these equations and then using an order-by-order perturbation in $\mid q - q_0 \mid$ one finds that the leading term in B_n is the one involving b and, of all the terms in eqn.(10), this constitutes the dominant contribution. The next-to-leading contributions in eqn.(10) are found to arise from terms involving A in A_n and C in C_n. Truncating the perturbation calculation at this order, we get

$$A = \frac{6\beta b_0}{(q^2 - 2\epsilon - 3q^2)} b^2 \qquad (13a),$$

$$C = \frac{3\beta b_0}{((q - 2 + 2cos2\phi)^2 - 2\epsilon - 3q^2)} b^2 \qquad (13b),$$

$$b^2 = \frac{((q - 2 - 2cos\phi)^2 - 2\epsilon - 3q^2)}{\gamma} \qquad (13c),$$

where

$$\gamma = \frac{36\beta^2 b_0^2}{(q^2 - 2\epsilon - 3q^2)} + \frac{18\beta^2 b_0^2}{((q - 2 + 2cos2\phi)^2 - 2\epsilon - 3q^2)} + 3\beta \qquad (14).$$

It is found that $\gamma > 0$ and hence there exist two 1-parameter families of invariant curves (with associated families of 2-tori) around the period-2 points with rotation numbers $\frac{\phi}{2\pi}$ where the two families correspond to $\hat{\phi} > \hat{\phi}_1$ and $\hat{\phi} < \hat{\phi}_2$ respectively, with

$$\hat{\phi}_{1,2} = \pm \frac{\sqrt{-2\epsilon/3}}{2sin\theta_0} \qquad (15).$$

The two families merge into a single family as q approaches q_0 from above and then detach from the period-2 orbit as a single one-parameter family of invariant curves (together with associated 2-tori) for $q < q_0$. In accordance with the results of ref.s [9] and [10], the secondary bifurcation can be described as a 'normal reversible Hopf bifurcation' (an alternative nomenclature is the Naimark-Sacker bifurcation, see [17]).

Fig.5 shows some of the invariant curves in 2-D projection, obtained on numerical iterations, on the (a) subthreshold and (b) superthreshold side (for details, see legend) of the secondary bifurcation. These iterations corroborate the results stated above relating to the secondary bifurcation associated with second order resonant collisions in families of 4D reversible maps. While the invariant curves in fig.(5a) belong to a so-called 'attached' family, those in fig.(5b) are members of what may be termed a 'detached' family of invariant curves (see sec. 5 below for an explanation).

IV. THIRD ORDER RESONANT COLLISIONS.

The 2-parameter family of 4D maps, written in the form of the following 4th order difference equation, describes a third order resonant collision at $q = \epsilon = 0$:

$$x_{n+2} + x_{n-2} - 2(q-1)(x_{n+1} + x_{n-1}) + (q^2 - 2q + 3 + \epsilon)x_n = \alpha x_n^2 + \beta x_n^3 \qquad (16),$$

where ϵ and q are small bifurcation parameters and, once again, the origin is a symmetric fixed point.

It is to be noted that the second degree non-linear term in eqn.(16) happens to be adequate in describing the bifurcation scenario for the third order resonant collision,

since third and higher degree terms are found to lead to no new bifurcation types while, for the second order collision, for which the linearisation of the map at $q = \epsilon = 0$ has a distinct Jordan block structure, the third degree term is necessary to describe the full scenario. Hence we set $\beta = 0$ in the following, though in our numerical iterations a non-zero value of β has been adopted.

The multipliers of the origin undergo a collision at $e^{\pm\,\frac{2\pi i}{3}}$ on the unit circle when $q = 0$ and $\epsilon = 0$ and, for small q, ϵ, bifurcating period-3 orbits exist close to the fixed point. A typical bifurcating period-3 orbit is of the form

$$\bar{x}_n = a + be^{\frac{2n\pi i}{3}} + b^* e^{-\frac{2n\pi i}{3}} \qquad (17),$$

where a and b can be obtained by substituing eqn.(17) in eqn.(16) and equating the coefficients of the constant term and of $e^{\frac{2n\pi i}{3}}$ respectively.

One can evaluate these constants perturbatively in terms of the bifurcation parameters. For the purpose of the present paper where we are interested only in describing the different bifurcation scenarios, we retain only the terms of the leading order in the bifurcation parameters.

Such a leading order calculation shows that, for any given q and ϵ there occur, apart from the trivial solution $a = b = 0$, two distinct non-trivial solutions for b, associated with a single solution for a. However, both the solutions are found to correspond to the *same* period-3 orbit when substituted in eqn.(17). A further observation is that the bifurcating orbit is independent of the signature of α, and we can chose

$$\alpha > 0$$

with no loss of generality. In summary, there occurs only one bifurcating period-3 orbit for each point in parameter space, and the values of a, b, describing this orbit are (with α chosen positive)

$$b = b^* = \frac{\epsilon + q^2}{\alpha} \qquad (18a),$$

$$a = \frac{2\alpha \mid b \mid^2}{(q-3)^2} \qquad (18b).$$

Linear stability analysis of the period-3 orbit can be done in essentially the same way as in the period-2 case, the multipliers ($\Lambda = \lambda^3$, where λ is a quasi-multiplier) being given by

$$(\Lambda + \frac{1}{\Lambda})^2 + A(\Lambda + \frac{1}{\Lambda}) + B = 0 \qquad (19),$$

where, up to $O(\epsilon^2)$ terms,

$$A = (-4 + 6q^2 - 6\epsilon) + (6\epsilon q - 2q^3) + \frac{8}{3}(\epsilon + q^2)^2 \qquad (20a),$$

$$B = (4 - 12q^2 + 12\epsilon) + (4q^3 - 12\epsilon q) - \frac{291}{9}(\epsilon + q^2)^2 \qquad (20b).$$

Thus, the multipliers of the period-3 orbit are obtained from the expression

$$\Lambda + \frac{1}{\Lambda} = 2 + 3\epsilon - 3q^2 \pm 6\sqrt{\epsilon^2 + q^4 + \epsilon\, q^2} \qquad (21).$$

It is easily seen from eqn.(21) that, for arbitrary choice of the parameters ϵ and q^2, one pair of multipliers of the period-3 orbit lies on the unit circle and the other on the real line, corresponding to the disposition $R3$ in fig.4. In other words, the bifurcating orbit is unstable in its domain of existence.

The bifurcation diagram following from these results is presented in fig.6. One finds that, for $\epsilon < 0$ (the subthreshold side of the bifurcation), the period-3 orbit merges with the fixed point for $\mid q^2 + \epsilon \mid \rightarrow 0$ while, for $\epsilon > 0$ (superthreshold side) the orbit is bounded away from the fixed point, the orbit being unstable in both the cases. This diagram corresponds to what has been termed a bifurcation of *type III* in the context of fourth order resonant collisions in ref.[15] (see next section).

In summary, while the bifurcation scenario for the second order resonant collision consists of two types of bifurcation (type I and type II of sec. III), only *one* type (analogous to type III of ref.[15]) is found to occur in the bifurcation scenario associated with the third order collision. In fig.7 we show the 2D projection of typical trajectories starting close to the period-3 orbit near the bifurcation, where one finds that the trajectories do diverge away from the orbit, in conformity with its unstable nature.

V. RESONANT COLLISIONS OF ORDER FOUR

In order to describe the bifurcation of period-4 orbits arising from a collision of multipliers at $\pm i$ ($k = 4$), we consider, as in ref.[15], the following two-parameter family of maps $A_{q,\epsilon}$, written in the form of a 4th order difference equation :

$$x_{n+2} + x_{n-2} + 4q(x_{n+1} + x_{n-1}) + (4q^2 + 2 + 4\epsilon)x_n = \beta(x_n^3 + \gamma x_{n+1}x_n x_{n-1}) \quad (22),$$

where $q,\ \epsilon$ are small bifurcation parameters and β and γ are control parameters which will be found in the sequel to govern the nature or the type of the bifurcation. These parameters will be assumed to satisfy the nondegeneracy conditions

$$\beta \neq 0 \qquad and \qquad \gamma^2 \neq 1 \qquad (23).$$

Quadratic terms in eqn.(22) have not been included since they are found not to affect the scenario of bifurcations involved. The origin is once again a symmetric fixed point of $A_{q,\epsilon}$, and the multipliers at the origin undergo a collision at $\pm i$ at $q = 0$ and ϵ=0. A typical bifurcating period-4 orbit of (22) is of the form

$$\bar{X}_n = (i)^n b_0 + (-i)^n b_0^* \qquad (24),$$

where b_0 is obtained by substituting eqn (24) in (22) and equating the coefficients of $(i)^n$ and $(-i)^n$. Two distinct types of bifurcating period-4 orbits are obtained in this way, which we denote by $O1$ and $O2$ respectively.

$$O1\ : \qquad b_0 = b_0^* = \rho_1 \qquad (25a),$$

$$with \qquad \rho_1^2 = \frac{\epsilon + q^2}{\beta} \qquad (25b).$$

$$O2 \;: \qquad b_0 = \rho_2(1+i) \qquad (26a),$$

$$with \qquad \rho_2^2 = \frac{\epsilon + q^2}{\beta(1-\gamma)} \qquad (26b).$$

As explained in [15], the two types of orbits have distinct symmetry properties under the reversing involutions pertaining to (22).

The linear stability analysis for the bifurcating period-4 orbits can be done in the same way as that for period-2 orbit from the appropriate variational equations. This has been presented in detail in ref.[15] and will not be repeated here. When the stability results of [15] are combined with eqn.s (25a) through (26b), one finds that the bifurcation scenario associated with the fourth order resonance involves *three* bifurcation types. We designate these as bifurcations of type I, II, and III, corresponding respectively to bifurcation diagrams of fig.s 8, 9, and 10. A brief description of these types, corresponding to different sets of values of the control parameters β and γ, is as follows.

Type I : $\beta > 0$, $\gamma < 1$

For $\gamma < -1$, on the subthreshold side ($\epsilon < 0$), both the bifurcating branches of period-4 orbits ($O1$ and $O2$) exist for $q > \sqrt{-\epsilon}$ and $q < -\sqrt{-\epsilon}$ (i.e., in regions $P1a$ and $P1b$ of the parameter space, refer to fig.1). In this situation $O1$ is linearly stable (disposition $R1$ of fig. 4) and $O2$ is unstable (disposition $R3$) (fig.8a). For $\epsilon = 0$, they merge together at the origin at $q = 0$(fig.8b) and for $\epsilon > 0$, they get detached from the origin with their respective stability types remaining unchanged (fig.8c). The situation remains essentially the same for $-1 < \gamma < 1$, with only the stability types getting interchanged (disposition $R3$ for $O1$ and $R1$ for $O2$).

Type II : $\beta < 0$, $\gamma < 1$

For $\epsilon < 0$, two branches exist simultaneously for $-\sqrt{-\epsilon} < q < \sqrt{-\epsilon}$, and are

globally connected (as the second bifurcation parameter q is varied), getting annihilated simultaneously as ϵ becomes positive.

Here we encounter once again the interesting situation that one of the two branches undergoes a change of stability at some critical value of q (for fixed $\epsilon < 0$) through a collision of Floquet multipliers on the unit circle near +1 (Fig.9). Thus, for $\gamma < 0$, the branch $O2$ is unstable (disposition $R3$) throughout the range $-\sqrt{|\epsilon|} < q < \sqrt{|\epsilon|}$ while, for the $O1$ solution, there occurs a transition from linearly stable (disposition $R1$) to unstable(disposition $R2$) configuration as q^2 crosses the threshold

$$q_1^2 = -\epsilon \frac{1 - \gamma + 2\sqrt{2}\sqrt{-1-\gamma}}{5 - \gamma - 2\sqrt{2}\sqrt{-1-\gamma}} \tag{27}$$

from above, due to a 'small angle' collision (see [19] for an explanation) of Floquet multipliers. As q^2 is made to decrease further, a second transition occurs at

$$q^2 = q_2^2 = -\epsilon \frac{1 - \gamma - 2\sqrt{2}\sqrt{-1-\gamma}}{5 - \gamma - 2\sqrt{2}\sqrt{-1-\gamma}} \qquad (28),$$

the disposition of the mutipliers changing over to $R4$ which, however, does not change the unstable character of the branch (this transition has not been shown in fig. 9). For $0 < \gamma < 1$, on the other hand, the $O1$ orbit is unstable (disposition $R3$) throughout the range $-\sqrt{|\epsilon|} < q < \sqrt{|\epsilon|}$, while for the $O2$ branch there occurs a transition from linearly stable($R1$) to unstable ($R2$) configuration as q^2 crosses the value

$$q_3^2 = -\epsilon \frac{1 + \sqrt{1-\gamma^2}}{2 - \gamma + \sqrt{1-\gamma^2}} \qquad (29),$$

from above, due to a small angle collision of the multipliers. As q^2 is made to decrease further, a second transition occurs as in the case of the $O1$ branch above.

Type III : $\gamma > 1$.

In this case one branch ($O1$ for $\beta > 0$) exists for $q > \sqrt{-\epsilon}$ and $q < -\sqrt{-\epsilon}$ (regions $P1a$ and $P1b$ of fig. 1) and the other branch ($O2$) exists for $-\sqrt{-\epsilon} < q < \sqrt{-\epsilon}$

(Fig.10a). Each of the branches is unstable (disposition $R3$) in its domain of existence. For $\epsilon > 0$ the branch $O1$ gets detached from the origin and $O2$ is annihilated (Fig.10c). The branches get interchanged if $\beta < 0$.

Thus, only for the type II collision near $\pm i$ (i.e., for $\beta < 0$ and $\gamma < 1$), there occurs a transition of a period-4 branch from linearly stable (disposition $R1$) to an unstable ($R2$) configuration through a small angle secondary collision. This is the co-dimension 1 collision discussed in ref.s [9,10]) and so the full co-dimension 2 collision studied in [19] is not relevant here. The transition in the disposition of multipliers from $R2$ to $R4$ referred to above again involves a co-dimension 1 collision, and will not be discussed here (see, e.g., [18], section 36).

The phase space dynamics in the vicinity of the secondary collision can once again be investigated by going beyond the linear terms. The analysis is essentially similar to the one outlined in case of the second order resonant collision and has been presented in detail in ref. [15]. The principal result in this respect is that *the secondary bifurcation is superthreshold or 'normal' in character* (see [9,10] for an explanation).

One can proceed further from this result and construct perturbatively the 'islands' of invariant curves surrounding the period-4 points in the vicinity of the secondary collision. Thus, on the subthreshold side of the bifurcation, each linearly stable period-4 point is surrounded by a family of invariant curves (forming an island, there being four such islands for the period-4 orbit under consideration), members of the family passing arbitrarily close to the period-4 point for rotation angles chosen sufficiently close to $\frac{2\pi}{4}$. This family persists on the superthreshold side of the bifurcation, but now detached from the period-4 point under consideration, i.e., there is now a lower bound to the distance of orbits belonging to this family from the period-4 point. These two types of families, with zero and non-zero lower bounds to distances from the fixed point or periodic point under consideration will be termed 'attached' and 'detached' families

respectively.

Attached families of invariant curves are commonly found to occur in reversible mappings around linearly stable fixed or periodic points (see ref. [15]), while there exists no standard result in the literature concerning *detached* families. As described in ref.s [10,11,15], such detached families are to be observed in the vicinities of 'normal' reversible Hopf bifurcations. We have already come across attached and detached families of invariant curves around period-2 points in sec. III.

Figures 11 and 12 represent results of numerical iterations with the family of maps $A_{q,\epsilon}$ (eqn. (22)) close to the secondary bifurcation. Figure 11a shows three invariant curves belonging to an attached family forming an island around one of the period-4 points of $O1$ on the subthreshold side of the bifurcation (for details, see legend) in a two-dimensional projection. Figure 11(b) on the other hand shows three invariant curves belonging to a detached family on the superthreshold side of the secondary bifurcation around the $O1$ orbit. Figures 12(a,b) show members belonging to corresponding families, on the subthreshold and superthreshold sides respectively, for the $O2$ branch (again,see legend, for details of each figure).

Two-dimensional projections of trajectories initiated very close to the two types of period-4 orbits ($O1$ and $O2$), on subthreshold and superthreshold sides of each of the three types of bifurcation have been presentend in fig.s 13, 14, and 15 respectively. In each figure the persistence of the trajectory signifies stability of that orbit, whereas a precipitate departure away from the orbit signifies its instability. Thus, fig. 13(a,b) represent 2D projections of trajectories initiated very close to the two types of period-4 orbits in type I bifurcation, where one finds that the trajectory around O1 persists near it forming an island, while one initiated near O2 departs away from it, following a path close to the separatrix.

Similarly, fig. 14(a-b) depict trajectories initiated near the O1 and O2 orbits in type

II bifurcation, now close to the *secondary* bifurcation. While fig. 14(a) corresponds to the subthreshold side of the secondary bifurcation, in which the trajectory initiated close to O1 forms an island and the one initiated close to O2 makes a circuit in the vicinity of the separatrix, fig. 14(b) corresponds to the *superthreshold* side, where trajectories are seen to depart away from *both* O1 and O2 orbits.

Fig. 15 presents similar trajectories for type III bifurcation in the fourth order resonant collision. As seen from fig. 10, there exists only one type of orbit ($O1$ *or* $O2$) for each specification of parameter values ($q,\ \epsilon$) on either side of the bifurcation ($\epsilon < 0, \epsilon > 0$), and that orbit is unstable. Fig. 15(a, b) show trajectories initiated close to the $O2$ and $O1$ orbits respectively for $\epsilon < 0$, and for two distinct values of q, and in each case the trajectory is seen to diverge away from the period-4 orbit concerned. Fig. 15(c), on the other hand depicts a similar trajectory for $\epsilon > 0$, where only the $O1$ orbit, now *detached* from the fixed point, survives.

VI. HIGHER ORDER RESONANT COLLISIONS.

Collisions of order five.

We consider fifth order resonant collisions with reference to a family of maps $A_{q,\epsilon}$ written in the form of a fourth order difference equation,

$$X_{n+2} + X_{n-2} - 4cos\phi_0(X_{n+1} + X_{n-1}) + (4cos^2\phi_0 + 2 + \epsilon)X_n = \alpha X_n^2 + \beta X_n^3 \quad (30),$$

where now $cos\phi_0 = cos\frac{2\pi}{5} + q/2$, ϵ and q being small bifurcation parameters as in the previous sections.

The multipliers of the origin undergo a collision at $e^{\frac{2\pi i}{5}}$ at $q = 0$ and $\epsilon = 0$. A typical bifurcating period-5 orbit is of the form

$$\bar{x}_n = a + be^{\frac{2n\pi i}{5}} + b^* e^{\frac{-2n\pi i}{5}} + ce^{\frac{4n\pi i}{5}} + c^* e^{\frac{-4n\pi i}{5}} \quad (31),$$

where $a, \ b, \ b^*, \ c, \ c^*$ can be obtained perturbatively by substitution in eq. (30).

In the leading order in ϵ and q^2, one finds

$$a_0 \approx \frac{2\alpha \mid b \mid^2}{\epsilon_0} \qquad (32a),$$

$$c \approx \frac{\alpha b^2}{\epsilon_2} \qquad (32b),$$

$$c^* \approx \frac{\alpha b^{*2}}{\epsilon_2} \qquad (32c),$$

and

$$\mid b \mid = \ \rho_0 \qquad (32d),$$

with

$$\rho_0^2 = \frac{\epsilon_1}{\gamma} \qquad (33a),$$

where

$$\gamma = 2\alpha^2 (\frac{2}{\epsilon_0} + \frac{1}{\epsilon_2}) + 3\beta \qquad (33b),$$

the parameters $\epsilon_0, \ \epsilon_1$, and ϵ_2 being given by

$$\epsilon_0 = \epsilon + 4(cos\phi_0 - 1)^2 \qquad (34a),$$

$$\epsilon_1 = \epsilon + q^2 \qquad (34b),$$

$$\epsilon_2 = \epsilon + 4(cos\phi_0 - cos\frac{4\pi}{5})^2 \qquad (34c),$$

respectively.

These equations tell us that, in the leading order of perturbation, there is only one period five orbit for any given value of the parameters ϵ and q. However, on going over to the next order of perturbation (i.e., to terms of the second degree in ϵ and q) one finds that there exist, in fact, *two* distinct types of period-5 orbits, which we designate as $O1$ and $O2$ respectively, distinguished by the value of the coefficient b in eq. (31) :

$$O1 \ : \qquad b = b^* = \rho_a \qquad (35a),$$

$$with \qquad \rho_a = \rho_0 + \rho_{1a} \qquad (35b).$$

$$O2 \ : \qquad b = \rho_b e^{\frac{\pi i}{5}} \qquad (36a),$$

$$with \qquad \rho_b = \rho_0 + \rho_{1b} \qquad (36b).$$

Where ρ_{1i} $(i \ = \ a, b)$ is given by

$$\rho_{1i} = -\frac{\rho_0^4 \chi cos4\phi_i}{2\epsilon cos\phi_i + 4\chi\rho_0^3 cos4\phi_i} \qquad (i = a, b) \qquad (37),$$

with

$$\phi_a \ = \ 0, \quad \phi_b \ = \ \frac{\pi}{5} \qquad 38(a, b),$$

and

$$\chi = \frac{\alpha^3}{\epsilon_2^2} + \frac{3\alpha\beta}{\epsilon_2} \qquad (39).$$

In other words, there is a degeneracy between the two types of period-5 orbits in the leading order of perturbation, which is removed in the next higher order. It is also apparent from eqn. (39) that the quadratic term in eqn. (30) is important in this removal of degeneracy.

Having obtained the perturbative expressions for the period-5 orbits it is a routine, if tedious, affair to follow trajectories initiated close to the orbits for various sets of parameter values, scanning across the parameter space. Perturbation calculations of the type resorted to in the case of lower order resonant collisions present increasing magnitudes of difficulty as one comes across resonant collisions of order 5 and above. Hence, in absence of a perturbative stability analysis of the orbits and a perturbative construction of the families of invariant curves close to these orbits, we invoke the

approach of numerical construction of trajectories close to the periodic orbits so as to have an idea of the bifurcation *scenarios* involved.

In the present instance, such an approach does seem to lead to a reliable picture, which we summarise as follows :

Period-5 orbits can bifurcate from a fixed point in a fifth order resonant collision in either of two ways that resemble the type I and type II bifurcations in a fourth order resonant collision. Thus, in type I bifurcation there exists a stable and an unstable orbit on either side of the bifurcation ($\epsilon < 0$, and $\epsilon > 0$), while in type II bifurcation, a pair of orbits exists only on the subthreshold side $\epsilon < 0$, there being no period-5 orbit on the superthreshold side. Of the members of the pair bifurcating from the fixed point for $\epsilon < 0$, one is always unstable, while the other undergoes a secondary bifurcation which is 'normal' in nature (see sec. V).

Fig. 16 presents trajectories initiated close to the period-5 orbits in a type I bifurcation for $\epsilon < 0$ (fig. 16a) and $\epsilon > 0$ (fig. 16b). One does find from the figure that the trajectory near one of the orbits belongs to an island of invariant curves, signifying that the orbit concerned is stable, while the trajectory near the other orbit moves away following a separatrix loop, and that essentially the same picture persists on the two sides of the bifurcation.

Fig. 17, on the other hand, depicts a secondary bifurcation associated with a type II bifurcation where one finds that trajectories on the subthreshold side of the secondary bifurcation (fig. 17a) are similar to those in the type I bifurcation while, on the superthreshold side (fig. 17b), trajectories move away from *both* the orbits, signifying that they are now unstable.

Collisions of order six.

Bifurcations of period-6 orbits can be studied with the family of maps $(A_{q,\epsilon})$ written

in the form

$$X_{n+2} + X_{n-2} - 2(1+q)(X_{n+1} + X_{n-1}) + (q^2 + 2q + 3 + \epsilon)X_n = \alpha X_n^2 + \beta X_n^3 \quad (40),$$

where, as before, ϵ and q are small bifurcation parameters, and α, β are control parameters.

As seen from eq. (40) the multipliers of the origin undergo a collision at $e^{\frac{2\pi i}{6}}$ at $q = 0$ and $\epsilon = 0$. A typical bifurcating period-6 orbit is of the form

$$\bar{x}_n = a + d(-1)^n + be^{\frac{2n\pi i}{6}} + b^* e^{\frac{-2n\pi i}{6}} + ce^{\frac{4n\pi i}{6}} + c^* e^{\frac{-4n\pi i}{6}} \quad (41),$$

where $a,\ d,\ b,\ b^*,\ c,\ c^*$ can be obtained perturbatively. In the leading order of perturbation in ϵ and q^2 one finds

$$a_0 \approx \frac{2\alpha |\ b\ |^2}{\epsilon_0} \quad (42a),$$

$$c_0 \approx \frac{\alpha b^2}{\epsilon_2} \quad (42b),$$

$$c_0^* \approx \frac{\alpha {b^*}^2}{\epsilon_2} \quad (42c),$$

$$d_0 \approx \frac{\delta(b^3 + {b^*}^3)}{\epsilon_3} \quad (42d),$$

and

$$|\ b\ | = \rho_0 \quad (42e),$$

with

$$\rho_0^2 = \frac{\epsilon_1}{\gamma} \quad (43a),$$

where

$$\gamma = 2\alpha^2 \left(\frac{2}{\epsilon_0} + \frac{1}{\epsilon_2}\right) + 3\beta \quad (43b),$$

$$\epsilon_0 = \epsilon + (q-1)^2 \quad (43c),$$

$$\epsilon_1 = \epsilon + q^2 \quad (43d),$$

$$\epsilon_2 = \epsilon + (q+2)^2 \qquad (43e),$$

$$\epsilon_3 = \epsilon + (q+3)^2 \qquad (43f),$$

$$\delta = \frac{2\alpha^2}{\epsilon_2} + \beta \qquad (43g).$$

Thus, there exists only one bifurcating orbit in the leading order of perturbation which, however is to be interpreted as just a degeneracy between distinct orbits that is lifted in the higher orders. Indeed, in the next order of perturbation, one finds that as in the case of a resonant collision of order five there exist two distinct types of period-6 orbits, which we describe as follows :

$$O1 \;\; : \qquad b = b^* = \rho_a \qquad (44a),$$

$$with \qquad \rho_a = \rho_0 + \rho_{1a} \qquad (44b),$$

$$O2 \;\; : \qquad b = \rho_b e^{\frac{\pi i}{6}} \qquad (45a),$$

$$with \qquad \rho_b = \rho_0 + \rho_{1b} \qquad (45b),$$

ρ_{1i} $(i \;=\; a, b)$ being given by

$$\rho_{1i} = -\frac{\rho_0^5(\chi_1 cos\phi_i + \chi_2 cos5\phi_i)}{2\epsilon cos\phi_i + 5\rho_0^4(\chi_1 cos\phi_i + \chi_2 cos5\phi_i)} \qquad (46),$$

where

$$\frac{\phi_a \;=\; 0, \quad \phi_b \;= \pi}{6} \qquad (47a,b),$$

$$\chi_1 = \frac{2\alpha^2\delta}{\epsilon_2\epsilon_3} + \frac{3\beta\delta}{\epsilon_3} + \frac{6\alpha^2\beta}{\epsilon_2^2} + \frac{12\alpha^2\beta}{\epsilon_0^2} + \frac{12\alpha^2\beta}{\epsilon_0\epsilon_2} \qquad (47c),$$

$$\chi_2 = \frac{2\alpha^2\delta}{\epsilon_2\epsilon_3} + \frac{3\beta\delta}{\epsilon_3} + \frac{3\alpha^2\beta}{\epsilon_2^2} \qquad (47d).$$

Once again, having obtained the perturbative expressions for the period-6 orbits, the dynamics in phase space close to these orbits can be studied through numerical iterations

in order to construct the bifurcation scenario. The result emerging from such an exercise is :

The scenario in the sixth order resonant collision is analogous to that in the fifth order collision, consisting of two types of bifurcation resembling in turn the type I and type II bifurcations associated with the fourth order collision. In particular, the type II bifurcation involves a secondary collision whereby one of the two period-6 orbits undergoes a transition from the linear stablility to instability.

Evidence for the two types of bifurcations in the family of maps given by eqn.(40) is presented in the form of fig.s 18 and 19, corresponding in significance to fig.s 16 and 17 respectively for the two types of bifurcation in the fifth order resonant collision (see legends for details).

These observations based on numerical iterations, together with the results of the previous sections lead to a conjecture concerning resonant collisions in families of reversible mappings presented in the next section.

VII. SUMMARY : A CONJECTURE.

In this paper we have considered resonant collisions of the multipliers at a symmetric fixed point of families of 4D reversible maps specified as in sec. II. A resonant collision has been defined in sec. I as a collision of multipliers on the unit circle at angles $\pm\frac{2\pi}{k}$ (k positive integer). The case $k = 1$ involves the bifurcation of *fixed points*, as distinct from the cases $k > 1$ where periodic orbits are found to bifurcate from the fixed point under consideration, and has been investigated from a numerical point of view in ref.[19]. The present paper addresses resonant collisions with $k > 1$. The families of maps we have considered for different specified values of k, while conforming to the criteria mentioned in sec. II, have all been chosen to be of the so-called de Vogelaere type (see, e.g., ref. [9]), these being relatively easy to handle analytically and

numerically (eqn.22, corresponding to $k = 4$, is an exception since the de Vogelaere form does not lead to the complete scenario in this case).

Collecting the results obtained from perturbative calculations and numerical iterations we formulate the following conjecture :

There exists a characteristic bifurcation scenario for each of the cases $k = 2, 3, 4$*, and a fourth and distinct scenario for* $k > 4$*.*

The scenario for $k = 2$ *involves two bifurcation types as depicted in fig.s 2 and 3, and in each of these there exists only one bifurcating period-2 orbit. This orbit undergoes a secondary bifurcation at a particular value of* q $(0 < q < \sqrt{(-\epsilon)})$*, with attached and detached families of invariant curves on the subthreshold and superthreshold sides of the bifurcation respectively.*

There is only one bifurcation type involved in the scenario for $k = 3$ *(fig. 6), the associated period-3 orbit being unstable on either side of the bifurcation. This bifurcation is analogous to one of the three types involved in the scenario for* $k = 4$*.*

Three bifurcation types are involved in the scenario for $k = 4$ *(fig.s 8, 9, 10). We have termed these type I, type II, and type III bifurcations respectively. In one of these (type II) there occurs a secondary bifurcation of one of the two associated period-4 orbits at two symmetrically located values of* q *for any given* $\epsilon < 0$*.*

The scenario for any $k > 4$ *involves two bifurcation types analogous respectively to types I and II for* $k = 4$ *(fig.s 8, 9).*

The bifurcating periodic orbits are associated with families of invariant curves which may be of either ‘attached’ or ‘detached’ type (see sec. V for explanation) corresponding to whether the orbit under consideration is linearly stable or is an unstable orbit on the superthreshold side of a secondary bifurcation. Our perturbative and numerical approach gives us a good idea of these families sufficiently close to the resonant collisions.

Additionally, one can talk of 2-dim *tori* in the immediate neighbourhoods of the

bifurcating orbits. Analogous to the families of invariant curves, these tori can also be of the ‘attached’ and ‘detached’ types. Thus, each of fig.s 20(a,b) presents a two-dimensional projection of a torus around a period-4 orbit, respectively on the subthreshold and superthreshold sides of the secondary bifurcation in a resonant collision leading to a type II bifurcation.

In other words, it is possible to have considerable information of the dynamics in phase space close to a resonant collision of multipliers of a symmetric fixed point of a family of 4D reversible mappings from the perturbative approach coupled with numerical iterations of the type presented in this paper. One hopes that this will help in the formulation of rigorous results in this largely uncharted area [20].

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FIGURE CAPTIONS.

Fig. 1 Regions in the $(\epsilon - q)$ parameter sapce divided by the curves (i) $\epsilon = 0$ and (ii) $q^2 + \epsilon = 0$.

Fig. 2 Schematic bifurcation diagram for Type I bifurcation ($\beta > 0$) of a period-2 orbit around a symmetric fixed point for (a) $\epsilon < 0$, (b) $\epsilon = 0$, (c) $\epsilon > 0$. For explanation, see text, Sec. III.

Fig. 3 Schematic bifurcation diagram for Type II bifurcation ($\beta < 0$) of a period-2 orbit around a symmetric fixed point for $\epsilon < 0$. For explanation, see text. Note the occurence of the secondary bifurcation. The superthreshold side ($\epsilon > 0$) is not shown in the figure since there exists no period-2 orbit on this side.

Fig. 4 Different possible disposition of the Floquet multipliers at a symmetric fixed point or periodic orbit for 4D reversible maps. For explanation, see text, Sec III.

Fig. 5 (a) 2-dim projection of a set of invariant curves around one of the period-2 points in second order resonant collision (type II bifurcation) on the subthreshold side of the secondary bifurcation, with $\epsilon = -10^{-5}$, $\beta = -1$ and $q = 2.585 \times 10^{-3}$, corresponding to $\hat{\phi} = 2.17 \times 10^{-3}$, 2.37×10^{-3} and 2.57×10^{-3} respectively. (b)Projection of a set of invariant curves around the period-2 point on the superthreshold side of the secondary bifurcation, with $q = -2.58 \times 10^{-3}$ for $\hat{\phi} = 1 \times 10^{-4}$, and 8×10^{-5} respectively, other parameters being the same as in (a).

Fig. 6 Schematic bifurcation diagram of period-3 bifurcation ; (a) $\epsilon < 0$, (b) $\epsilon = 0$, (c) $\epsilon > 0$.

Fig. 7 2-dim projection of a trajectory initiated close to a period-3 point near the resonant collision with $\beta = 1$, $\alpha = 1$ corresponding to (a) $\epsilon = -1 \times 10^{-4}$, $q = 8 \times 10^{-3}$

and $\delta = 1 \times 10^{-9}$, and (b) $\epsilon = 1 \times 10^{-5}$, $q = 8 \times 10^{-3}$ and $\delta = 1 \times 10^{-9}$; here and in the subsequent figures the parameter δ specifies the initial condition for the trajectory; we first calculate period-3 orbit ($\bar{X}_n$) from eqn.(17) and then take the initial condition of the trajectory as $X_n = \bar{X}_n + n(-1)^n \delta$ ($n = 1, 2, 3, 4$).

Fig. 8 Schematic bifurcation diagram for Type I bifurcation ($\beta > 0$, $\gamma < 1$) of period-4 orbits around a symmetric fixed point for (a) $\epsilon < 0$, (b) $\epsilon = 0$, (c) $\epsilon > 0$. The labels $O1$ and $O2$ are shown for $\gamma < 0$ while for $0 < \gamma < 1$ they should be interchanged. For explanation, see text, sec.VI.

Fig. 9 Schematic bifurcation diagram for Type II bifurcation ($\beta < 0$, $\gamma < 1$). Only the subthreshold situation ($\epsilon < 0$) is shown. The orbits shrink to the fixed point at $\epsilon = 0$, and cease to exist for $\epsilon > 0$. The labels $O1$ and $O2$ are shown for $\gamma < 0$ while for $0 < \gamma < 1$ they should be interchanged.

Fig. 10 Schematic bifurcation diagram of Type-III bifurcation ($\gamma > 1$); (a) $\epsilon < 0$, (b) $\epsilon = 0$, (c) $\epsilon > 0$. The labels $O1$ and $O2$ are shown for $\beta > 0$ while for $\beta < 0$ they should be interchanged; see text, sec.VI.

Fig. 11 (a) 2-dim projection of a set of invariant curves around one of the period-4 points of the $O1$ orbit on the subthreshold side of the secondary bifurcation, with $\epsilon = -10^{-5}$, $\beta = -1$, $\gamma = -3.02$ and $q = -2.585 \times 10^{-3}$ corresponding to $\hat{\phi} = 1.173 \times 10^{-4}$, 1.178×10^{-4} and 1.183×10^{-4} respectively. (b) Projection of a set of invariant curves around one of the period-4 points of the 01 orbit on the superthreshold side of the secondary bifurcation, with $q = -2.5835 \times 10^{-3}$ for $\hat{\phi} = 1 \times 10^{-5}$, 3×10^{-5} and 5×10^{-5} respectively, other parameters being the same as in (a).

Fig. 12 (a) 2-dim projection of a set of invariant curves around one of the period-4 points of the $O2$ orbit on the subthreshold side of the secondary bifurcation, with

$\epsilon = -10^{-5}$, $\beta = -1$, $\gamma = -10^{-2}$ and $q = -2.578\times10^{-3}$ corresponding to $\hat{\phi} = 6.99\times10^{-5}$, 7.02×10^{-5} and 7.05×10^{-5} respectively. (b) Projection of a set of invariant curves around one of the period-4 points of the $O2$ orbit on the superthreshold side of the secondary bifurcation, with $q = -2.577\times10^{-3}$ for $\hat{\phi} = 1\times10^{-5}$, 2.2×10^{-5} and 3.4×10^{-5} respectively, other parameters being the same as in (a).

Fig. 13 2-dim projection of two trajectories starting from close to the period-4 orbits ($O1$ and $O2$) in type I bifurcation near the resonant collision with $\beta = 1$, $\gamma = 0$ and $q = 2\times10^{-2}$, corresponding to (a) $\epsilon = -1\times10^{-4}$, $\delta = 1\times10^{-4}$ for $O2$ and $\delta = 0$ for $O1$ orbit and (b) $\epsilon = 1\times10^{-4}$, $\delta = 1\times10^{-4}$ for $O2$ and $\delta = 0$ for $O1$ orbit; we first calculate period-4 orbit ($\bar{X}_n$) from eqn.(24) and then take the initial condition of the trajectory as $X_n = \bar{X}_n + \delta$ (δ being taken nonzero only for $n = 4$).

Fig. 14 2-dim projection of two trajectories starting from close to the period-4 orbits ($O1$ and $O2$) in type II bifurcation near the *secondary* bifurcation with $\beta = -1$, $\gamma = 0$ and $\epsilon = -1\times10^{-4}$, corresponding to (a) $q = 9.5\times10^{-3}$, $\delta = 5\times10^{-6}$ for $O2$ and $\delta = 1\times10^{-6}$ for $O1$ orbit and (b) $q = 8.1\times10^{-3}$, $\delta = 1\times10^{-6}$ for $O2$ and $\delta = 1\times10^{-6}$ for $O1$ orbit ; we first calculate period-4 orbit ($\bar{X}_n$) from eqn.(24) and then take the initial condition of the trajectory as $X_n = \bar{X}_n + \delta$ (δ being taken nonzero only for $n = 4$).

Fig. 15 2-dim projection of a trajectory starting from close to the period-4 orbit ($O1$ or $O2$) in type III bifurcation near the bifurcation point with $\beta = 1$, $\gamma = 2$ corresponding to (a) $\epsilon = -1\times10^{-4}$, $q = -9\times10^{-3}$ and $\delta = 2.5\times10^{-6}$ for $O2$, (b) $\epsilon = -1\times10^{-4}$, $q = -1.2\times10^{-2}$ and $\delta = -1\times10^{-6}$ for $O1$ and (c) $\epsilon = 1\times10^{-5}$, $q = -1.2\times10^{-2}$ and $\delta = -1\times10^{-8}$ for $O1$ orbit; we first calculate period-4 orbit ($\bar{X}_n$) from eqn.(24) and then take the initial condition of the trajectory as $X_n = \bar{X}_n + n(-1)^n\delta cos(n\frac{\pi}{4})$.

Fig. 16 2-dim projection of two trajectories starting from close to the period-5 orbits

($O1$ and $O2$) in type I bifurcation near the bifurcation point with $\beta = -.5$, $\alpha = 1$ and $q = 6 \times 10^{-2}$ corresponding to (a) $\epsilon = -1 \times 10^{-4}$ $\delta = 1 \times 10^{-4}$ for $O2$ and $\delta = 0$ for $O1$ orbit and (b) $\epsilon = 1 \times 10^{-4}$, $\delta = 5 \times 10^{-5}$ for $O2$ and $\delta = 1 \times 10^{-6}$ for $O1$ orbit; we first calculate period-5 orbit ($\bar{X}_n$) from eqn.(31) and then take the initial condition of the trajectory as $X_n = \bar{X}_n + \delta$ (δ being taken nonzero only for $n = 4$).

Fig.17 2-dim projection of two trajectories starting from close to the period-5 orbits ($O1$ and $O2$) in type II bifurcation near the *secondary* bifurcation with $\beta = -2$, $\alpha = 1$ and $\epsilon = -5\times10^{-3}$ corresponding to (a) $q = 6\times10^{-2}$ $\delta = 5\times10^{-5}$ for $O2$ and $\delta = 1\times10^{-4}$ for $O1$ orbit and (b) $q = 4.806 \times 10^{-2}$, $\delta = 0$ for $O2$ and $O1$ orbit ; we first calculate period-5 orbit ($\bar{X}_n$) from eqn.(31) and then take the initial condition of the trajectory as $X_n = \bar{X}_n + \delta$ (δ being taken nonzero only for $n = 4$).

Fig. 18 2-dim projection of two trajectories starting from close to the period-6 orbits ($O1$ and $O2$) in type I bifurcation near the bifurcation point with $\beta = 1$, $\alpha = 1$ and $q = -2 \times 10^{-1}$ corresponding to (a) $\epsilon = -1 \times 10^{-2}$ $\delta = 1 \times 10^{-4}$ for $O1$ and $\delta = 0$ for $O2$ orbit and (b) $\epsilon = 1 \times 10^{-3}$, $\delta = 2 \times 10^{-4}$ for $O1$ and $\delta = 0$ for $O2$ orbit; we first calculate period-5 orbit ($\bar{X}_n$) from eqn.(41) and then take the initial condition of the trajectory as $X_n = \bar{X}_n + 2n(-1)^n\delta cos(2n\phi)$ (ϕ=0 for $O1$ and $\frac{\pi}{6}$ for $O2$).

Fig. 19 2-dim projection of two trajectories starting from close to the period-6 orbits ($O1$ and $O2$) in type II bifurcation near the *secondary* bifurcation with $\beta = -4$, $\alpha = 1$ and $\epsilon = -1 \times 10^{-2}$ corresponding to (a) $q = -6.8 \times 10^{-2}$ $\delta = -1.6 \times 10^{-6}$ for $O2$ and $\delta = 1 \times 10^{-6}$ for $O1$ orbit and (b) $q = -6.2 \times 10^{-2}$, $\delta = 0$ for $O2$ and $O1$ orbit ; we first calculate period-6 orbit ($\bar{X}_n$) from eqn.(41) and then take the initial condition of the trajectory as $X_n = \bar{X}_n + 2n(-1)^n\delta cos(2n\phi)$. ($\phi$=0 for $O1$ and $\frac{\pi}{6}$ for $O2$). The inset shows one of the stable period-6 orbit surrounded by the sepratrix of the unstable period-6 point.

Fig. 20 (a) 2-dim projection of a trajectory around one of the period-4 points of the $O1$ orbit on the subthreshold side of the secondary bifurcation, with $\epsilon = -10^{-5}$, $\beta = -1$, $\gamma = -3.02$ and $q = -2.585 \times 10^{-3}$; we first calculate an initial condition (X_1, X_2, X_3, X_4) corresponding to one member belonging to a family of invariant curves, with $\hat{\phi} = 1.18 \times 10^{-4}$ (see ref.s [11,15] for explanation); a slightly different initial condition is then taken with X_1, X_2, X_3 unchanged and X_4 replaced by $X_4 + \delta$ ($\delta = 3 \times 10^{-10}$) and iterations performed; the trajectory winds on a 2-torus around the invariant curve. (b) A 2-torus on the superthreshold side near a period-4 point of the $O1$ orbit ; with $\epsilon = -10^{-5}$, $\beta = -1$, $\gamma = -10^{-2}$ and $q = -2.577 \times 10^{-3}$, $\hat{\phi} = 10^{-5}$, $\delta = 10^{-9}$ (see legend for (a)).